INTRODUCTION TO METRIC SPACES WITH DILATIONS

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Abstract. This paper gives a short introduction into the metric theory of spaces with dilations.

1. Introduction

Metric spaces with dilations were introduced in [4] under the name of "dilatation structures", then studied in a series of papers [5] [6] [7]. Very recently, in [19], [20], the same object has been named "(quasi)metric space with dilations". In the mentioned papers the authors extend the results from [4] to quasimetric spaces. We

Date: This version: 15.07.2010.
1991 *Mathematics Subject Classification.* 51K10, 53C17, 53C23.

shall keep here this double denomination dilatation structure - metric space with dilations.

Topological spaces with dilations were studied for the first time to my knowledge in the paper [3]. In the paper [8] it is proved that the algebraic properties of spaces with dilations are not based on metric notions, but in fact they hold for uniform spaces. Thus the generalization of Selivanova and Vodopyanov is not surprising at all, because quasimetric spaces are uniform topological spaces and this is all we need in order to deduce these mentioned algebraic properties. Another line of generalization was proposed in [9], where normed groupoids and specific deformations of those were introduced. A particular case is that of a trivial normed groupoid with a deformation induced by a dilatation structure.

Finally, in the paper [10] we introduced length metric spaces with dilations (length dilatation structures) and proved that regular sub-riemannian spaces can be seen as such length dilatation structures. In the case of length metric spaces with dilations we have to work with length functionals and study the gamma-convergence, or variational convergence, of length functionals, thus generalizing results obtained by Buttazzo, De Pascale, and Fragalà in [13], or Venturini [21].

In this paper I give a short introduction into these subjects, which could serve as a basis for understanding more specialized results.

In my opinion spaces with dilations could become a topic of intense studies. Indeed, many examples studied in analysis in metric spaces are in fact spaces with dilations and it seems that this supplementary algebraic-geometric structure which was recently identified could be a valuable tool for developing differential calculus or geometric measure theory in such spaces. For the moment this subject has not been explored in combination with measure theory (for example on metric measured spaces, or in relation with optimal transportation). But it seems reasonable to expect that new results await just around the corner.

2. Metric spaces, distances, norms

Definition 1. *A metric space (X, d) is a set X endowed with a distance function $d : X \times X \to [0, +\infty)$. In the metric space (X, d), the distance between two points $x, y \in X$ is $d(x, y) \geq 0$. The distance d satisfies the following axioms:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,
(ii) *(symmetry) for any* $x, y \in X$ $d(x, y) = d(y, x)$,
(iii) *(triangle inequality) for any* $x, y, z \in X$ $d(x, z) \leq d(x, y) + d(y, z)$.

The ball of radius $r > 0$ and center $x \in X$ is the set

$$B(x, r) \, = \, \{ y \in X \, : \, d(x, y) < r \} \quad .$$

Sometimes we shall use the notation $B_d(x, r)$ for the ball of center x and radius r with respect to the distance d, in order to emphasize the dependence on the distance d. Any metric space (X, d) is endowed with the topology generated by balls. The

notations $\bar{B}(x,r)$ and $\bar{B}_d(x,r)$ are used for the closed ball centered at x, with radius r.

A pointed metric space (X,x,d) is a metric space (X,d) with a chosen point $x \in X$.

The notion of a metric space is not very old: it has been introduced by Fréchet in the paper [Sur quelques points du calcul fonctionnel, *Rendic. Circ. Mat. Palermo 22* (1906), 1-74].

2.1. **Metric spaces, normed groups and normed groupoids.** An obvious example of a metric space is $\mathbb{R}^n$ endowed with an euclidean distance, that is with a distance function induced by an euclidean norm:

$$d(x,y) \,=\, \|x-y\| \quad .$$

In fact any normed vector space can be seen as a metric space. In order to define a distance from a norm, in a normed vector space, we only need the norm function and the abelian group structure of the vector space. (Later in this paper, he multiplication by scalars will provide us with the first example of a metric space with dilations). This leads us to the introduction of normed groups. Let us give, in increasing generality, the definition of a normed group, then the definition of a normed groupoid.

Definition 2. *A normed group (G,ρ) is a pair formed by:*

- *a group G, with the operation $(x,y) \in G \times G \mapsto xy$, inverse denoted by $x \in G \mapsto x^{-1}$ and neutral element denoted by e,*
- *a norm function $\rho : G \to [0,+\infty)$, which satisfies the following axioms:*
  - (i) *$\rho(x) = 0$ if and only if $x = e$,*
  - (ii) *(symmetry) for any $x \in G$ $\rho(x^{-1}) = \rho(x)$,*
  - (iii) *(sub-additivity) for any $x, y \in G$ $\rho(xy) \leq \rho(x) + \rho(y)$.*

Proposition 3. *Any normed group (G,ρ) can be seen as a metric space, with any of the distances*

$$d_L(x,y) \,=\, \rho(x^{-1}y) \quad , \quad d_R(x,y) \,=\, \rho(xy^{-1}) \quad .$$

The function d_L is left-*invariant, i.e. for any $x,y,z \in G$ we have $d_L(zx,zy) = d_L(x,y)$. Similarly d_R is* right-*invariant, that is for any $x,y,z \in G$ we have $d_R(xz,yz) = d_R(x,y)$.*

Proof. It suffices to give the proof for the distance d_L. Indeed, the first axiom of a distance is a consequence of the first axiom of a norm, the symmetry axiom for distances is a consequence of the symmetry axiom of the norm and the triangle inequality comes from the group identity

$$x^{-1}z \,=\, \left(x^{-1}y\right)\left(y^{-1}z\right)$$

(which itself is a consequence of the associativity of the group operation and of the existence of inverse) and from the sub-additivity of the norm. The left-invariance of d_L comes from the group identity $(zx)^{-1}(zy) = x^{-1}y$. □

Groupoids are generalization of groups. A groupoid can be seen as a small category such that any arrow is invertible. Alternatively, if we look at the set of arrows of such a category, it is a set with a partially defined binary operation and a unary operation (the inverse function), which satisfy several properties. A norm is then a function defined on the set of arrows of a groupoid, with properties similar with the ones of a norm over a group. This is the definition which we give further.

Definition 4. *A normed groupoid (G, ρ) is a pair formed by:*

- *a groupoid G, which is a set with two operations $inv : G \to G$, $m : G^{(2)} \subset G \times G \to G$, which satisfy a number of properties. With the notations $inv(a) = a^{-1}$, $m(a, b) = ab$, these properties are: for any $a, b, c \in G$*
  - (i) *if $(a, b) \in G^{(2)}$ and $(b, c) \in G^{(2)}$ then $(a, bc) \in G^{(2)}$ and $(ab, c) \in G^{(2)}$ and we have $a(bc) = (ab)c$,*
  - (ii) *$(a, a^{-1}) \in G^{(2)}$ and $(a^{-1}, a) \in G^{(2)}$,*
  - (iii) *if $(a, b) \in G^{(2)}$ then $abb^{-1} = a$ and $a^{-1}ab = b$.*

  The set $X = Ob(G)$ is formed by all products $a^{-1}a$, $a \in G$. For any $a \in G$ we let $\alpha(a) = a^{-1}a$ and $\omega(a) = aa^{-1}$.
- *a norm function $d : G \to [0, +\infty)$ which satisfies the following axioms:*
  - (i) *$d(g) = 0$ if and only if $g \in Ob(G)$,*
  - (ii) *(symmetry) for any $g \in G$, $d(g^{-1}) = d(g)$,*
  - (iii) *(sub-additivity) for any $(g, h) \in G^{(2)}$, $d(gh) \leq d(g) + d(h)$,*

If $Ob(G)$ is a singleton then G is just a group and the previous definition corresponds exactly to the definition 2 of a normed group. As in the case of normed groups, normed groupoids induce metric spaces too.

Proposition 5. *Let (G, d) be a normed groupoid and $x \in Ob(G)$. Then the space $(\alpha^{-1}(x), d_x)$ is a metric space, with the distance d_x defined by: for any $g, h \in G$ with $\alpha(g) = \alpha(h) = x$ we have $d_x(g, h) = d(gh^{-1})$.*

Therefore a normed groupoid can be seen as a disjoint union of metric spaces

$$G = \bigcup_{x \in Ob(G)} \alpha^{-1}(x) \quad , \tag{1}$$

with the property that right translations in the groupoid are isometries, that is: for any $u \in G$ the transformation

$$R_u : \alpha^{-1}(\omega(u)) \to \alpha^{-1}(\alpha(u)) \quad , \quad R_u(g) = gu$$

has the property for any $g, h \in \alpha^{-1}(\omega(u))$

$$d_{\omega(u)}(g, h) = d_{\alpha(u)}(R_u(g), R_u(h)) \quad .$$

Proof. We begin by noticing that if $\alpha(g) = \alpha(h)$ then $(g, h^{-1}) \in G^{(2)}$, therefore the expression gh^{-1} makes sense. The rest of the proof of the first part of the proposition is identical with the proof of the previous proposition.

For the proof of the second part of the proposition remark first that R_u is well defined and that

$$R_u(g)\,(R_u(h))^{-1} \,=\, gh^{-1} \quad .$$

Then we have:

$$d_{\alpha(u)}(R_u(g), R_u(h)) \,=\, d\left(R_u(g)\,(R_u(h))^{-1}\right) \,=$$
$$=\, d(gh^{-1}) \,=\, d_{\omega(u)}(g,h) \quad .$$

□

Therefore normed groupoids provide examples of (disjoint unions of) metric spaces. Are there metric spaces more general than these? No, in fact we have the following.

Proposition 6. *Any metric space can be constructed from a normed groupoid, as in proposition 5. Precisely, let (X, d) be a metric space and consider the* trivial groupoid *$G = X \times X$ with multiplication*

$$(x,y)(y,z) \,=\, (x,z)$$

and inverse $(x,y)^{-1} = (y,x)$. Then (G,d) is a normed groupoid and moreover any component of the decomposition (1) of G is isometric with (X,d).

Conversely, if $G = X \times X$ is the trivial groupoid associated to the set X and d is a norm on G then (X,d) is a metric space.

Proof. We begin by noticing that $\alpha(x,y) = (y,y)$, $\omega(x,y) = (x,x)$, therefore $Ob(G) = \{(x,x) : x \in X\}$ can be identified with X by the bijection $(x,x) \mapsto x$. Moreover, for any $x \in X$ we have

$$\alpha^{-1}((x,x)) \,=\, X \times \{x\} \quad .$$

Because $d : X \times X \to [0,+\infty)$ and $G = X \times X$ it follows that $d : G \to [0,+\infty)$. We have to check the properties of a norm over a groupoid. But these are straightforward. The statement (i) ($d(x,y) = 0$ if and only if $(x,y) \in Ob(G)$) is equivalent with $d(x,y) = 0$ if and only if $x = y$. The symmetry condition (ii) is just the symmetry of the distance: $d(x,y) = d(y,x)$. Finally the sub-additivity of d seen as defined on the groupoid G is equivalent with the triangle inequality:

$$d((x,y)(y,z)) \,=\, d(x,z) \,\leq\, d(x,y) + d(y,z) \quad .$$

In conclusion (G,d) is a normed groupoid if and only if (X,d) is a metric space.

For any $x \in X$ the distance $d_{(x,x)}$ on the space $\alpha^{-1}((x,x))$ has the expression:

$$d_{(x,x)}((u,x),(v,x)) \,=\, d((u,x)(v,x)^{-1}) \,=\, d((u,x)(x,v)) \,=\, d(u,v)$$

therefore the metric space $(\alpha^{-1}((x,x)), d_{(x,x)})$ is isometric with (X,d) by the isometry $(u,x) \mapsto u$, for any $u \in X$. □

In conclusion normed groups give particular examples of metric spaces and metric spaces are particular examples of normed groupoids. For this reason normed groups make good examples of metric spaces. It is also interesting to extend the theory of metric spaces to normed groupoids (other than trivial normed groupoids). This is done in [9].

2.2. **Gromov-Hausdorff distance.** For this subject see [2] (Section 7.4), [15] (Chapter 3) and [16]. We start the presentation by a discussion about maps and microscopes.

Imagine that the metric space (X, d) represents some part of the world, like the collection of the cities in a country. We also need a injective function $Name : X \to A$, which associated to any $x \in X$ the object $Name(x)$ which represents the name of the place x (the set A is a collection of names). (It seems that the function $Name$ is not really necessary for this process. Indeed, in this abstract mathematical description we use statements like "to $x \in X$ we associate $y \in Y$", so the letters x, y are just generic names. In conclusion, in the following we may take $Name(x) = x$ without altering the discussion).

The distance between two places called $Name(x)$ and $Name(y)$ is equal to $d(x, y)$. Suppose that we want to mathematically describe what is a map of the collection $(X, d, Name)$ in the metric space (Y, d'), at the scale $\varepsilon > 0$. For example (Y, d') might represent a printed map of the region $(X, d, Name)$. For the moment we take $\varepsilon = 1$, meaning that we want to make a map of $(X, d, Name)$, at the scale 1:1, in (Y, d').

We might say that such a map of $(X, d, Name)$ in (Y, d') is in fact a relation $\rho \subset X \times Y$. To the place named $Name(x)$ is associated the set of points $\{y \in Y : (x, y) \in \rho\}$. We then decorate our map ρ with names by defining *a relation* $Name' \subset Y \times A$, given like this: $(x, y) \in \rho$ if and only if $(y, Name(x)) \in Name'$. At this point we would like that the map ρ preserves the distances up to a precision μ.

Let us simplify our notations concerning relations. For any relation $\rho \subset X \times Y$ we shall write $\rho(x) = y$ if $(x, y) \in \rho$. Therefore we may have $\rho(x) = y$ and $\rho(x) = y'$ with $y \neq y'$, if $(x, y) \in f$ and $(x, y') \in f$.

The domain of the relation ρ is the set $dom\ \rho \subset X$ such that for any $x \in dom\ \rho$ there is $y \in Y$ with $\rho(x) = y$. The image of ρ is the set of $im\ \rho \subset Y$ such that for any $y \in im\ \rho$ there is $x \in X$ with $\rho(x) = y$. By convention, when we write that a statement $R(f(x), f(y), ...)$ is true, we mean that $R(x', y', ...)$ is true for any choice of $x', y', ...,$ such that $(x, x'), (y, y'), ... \in f$.

When we make a map ρ we are not really measuring the distances between all points in X, then consider a bijection from X to Y. What we do is that first we take, for a number $\mu > 0$, a collection $M \subset X$ of points in X which is μ-dense in (X, d).

Definition 7. *A subset $M \subset X$ of a metric space (X, d) is μ-dense in X if for any $u \in X$ there is $x \in M$ such that $d(x, u) \leq \mu$.*

After measuring (or using other means to deduce) the distances $d(x', x")$ between all pairs of points in M (we may have several values for the distance $d(x', x"))$, we try to represent the collection of these distances in (Y, d'). Therefore we pick a subset $M' \subset Y$, which is μ-dense in (Y, d'), maybe in order to spare the material of this expensive 1:1 map. Then we associate to any $x \in M$ one or several points $y \in M'$ such that for any two points $x_1, x_2 \in M$ and for any choice of points $y_1, y_2 \in M'$, in correspondence with x_1, x_2 respectively, the distances $d(x_1, x_2)$ and $d'(y_1, y_2)$ differ by μ at most. The association to any point $x \in M$ of a point $y \in M'$ is the relation ρ, with domain M and image M'.

The infimum of all $\mu > 0$ for which such a map ρ is possible represents the greatest precision of making a map of (X, d) in (Y, d').

This infimum is in general not equal to zero. We may treat symmetrically the metric spaces (X, d) and (Y, d') and ask for the infimum of all μ such that (X, d) admits a map in (Y, d') with precision μ and (Y, d') admits a map in (X, d) with precision μ. This μ is called the Gromov-Hausdorff distance between the metric spaces (X, d) and (Y, d'). This distance can be also infinite if for any μ we cannot have a map ρ associated.

We shall use also the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a positive function such that $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$.

The definition of the Gromov-Hausdorff distance for pointed metric spaces is the following.

Definition 8. *Let (X_i, d_i, x_i), $i = 1, 2$, be a pair of locally compact pointed metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X_1 \times X_2$ such that*

1. *dom ρ is μ-dense in X_1,*
2. *im ρ is μ-dense in X_2,*
3. *$(x_1, x_2) \in \rho$,*
4. *for all $x, y \in$ dom ρ we have*

$$| \, d_2(\rho(x), \rho(y)) - d_1(x, y) \, | \leq \mu \tag{2}$$

The Gromov-Hausdorff distance between (X_1, x_1, d_1) and (X_2, x_2, d_2) is the infimum of admissible numbers μ.

As introduced in definition 8, the Gromov-Hausdorff (GH) distance is not a true distance, because the GH distance between two isometric pointed metric spaces is equal to zero. In fact the GH distance induces a distance on isometry classes of pointed metric spaces (which are not far apart). (The isometry class $[X, d_X, x]$ of the pointed metric space (X, d_X, x), is the class of spaces (Y, d_Y, y) such that it exists an isometry $f : X \to Y$ with the property $f(x) = y$.)

Indeed, if two pointed metric spaces are isometric then the Gromov-Hausdorff distance equals 0. The converse is also true in the class of compact (pointed) metric spaces [15] (Proposition 3.6).

Moreover, if two of the isometry classes $[X, d_X, x]$, $[Y, d_Y, y]$, $[Z, d_Z, z]$ have (representants with) diameter at most equal to 3, then the triangle inequality is true.

We shall use this distance and the induced convergence for isometry classes of the form $[X, d_X, x]$, with $diam\ X \ \leq 5/2$.

2.3. **Metric profiles. Metric tangent space.** We shall denote by CMS the set of isometry classes of pointed compact metric spaces. The distance on this set is the Gromov distance between (isometry classes of) pointed metric spaces and the topology is induced by this distance.

To any locally compact metric space we can associate a metric profile [11, 12].

Definition 9. *The metric profile associated to the locally metric space* (M, d) *is the assignment (for small enough* $\varepsilon > 0$*)*

$$(\varepsilon > 0, \ x \in M) \ \mapsto \ \mathbb{P}^m(\varepsilon, x) = \left[\bar{B}(x,1), \frac{1}{\varepsilon} d, x\right] \in CMS$$

We can define a notion of metric profile regardless to any distance.

Definition 10. *A metric profile is a curve* $\mathbb{P} : [0, a] \to CMS$ *such that*

(a) *it is continuous at* 0*,*
(b) *for any* $b \in [0, a]$ *and* $\varepsilon \in (0, 1]$ *we have*

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x_b)) \ = \ O(\varepsilon)$$

The function $\mathcal{O}(\varepsilon)$ *may change with* b*. We used the notations*

$$\mathbb{P}(b) = [\bar{B}(x,1), d_b, x_b] \quad \text{and} \quad \mathbb{P}^m_{d_b}(\varepsilon, x) = \left[\bar{B}(x,1), \frac{1}{\varepsilon} d_b, x_b\right]$$

The metric profile is nice if

$$d_{GH}\left(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)\right) = O(b\varepsilon)$$

Imagine that $1/b$ represents the magnification on the scale of a microscope. We use the microscope to study a specimen. For each $b > 0$ the information that we get is the table of distances of the pointed metric space $(\bar{B}(x,1), d_b, x_b)$.

How can we know, just from the information given by the microscope, that the string of "images" that we have corresponds to a real specimen? The answer is that a reasonable check is the relation from point (b) of the definition of metric profiles 10.

Really, this point says that starting from any magnification $1/b$, if we further select the ball $\bar{B}(x, \varepsilon)$ in the snapshot $(\bar{B}(x,1), d_b, x_b)$, then the metric space $(\bar{B}(x,1), \frac{1}{\varepsilon} d_b, x_b)$ looks approximately the same as the snapshot $(\bar{B}(x,1), d_{b\varepsilon}, x_b)$. That is: further magnification by ε of the snapshot (taken with magnification) b is roughly the same as the snapshot $b\varepsilon$. This is of course true in a neighbourhood of the base point x_b.

The point (a) from the Definition 10 has no other justification than Proposition 14 in next subsection.

We rewrite definition 8 with more details, in order to clearly understand what is a metric profile. For any $b \in (0, a]$ and for any $\mu > 0$ there is $\varepsilon(\mu, b) \in (0, 1)$ such

that for any $\varepsilon \in (0, \varepsilon(\mu, b))$ there exists a relation $\rho = \rho_{\varepsilon,b} \subset \bar{B}_{d_b}(x_b, \varepsilon) \times \bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$ such that

1. *dom* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_b}(x_b, \varepsilon)$,
2. *im* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$,
3. $(x_b, x_{b\varepsilon}) \in \rho_{\varepsilon,b}$,
4. for all $x, y \in$ *dom* $\rho_{\varepsilon,b}$ we have

$$\left| \frac{1}{\varepsilon} d_b(x, y) - d_{b\varepsilon}\left(\rho_{\varepsilon,b}(x), \rho_{\varepsilon,b}(y)\right) \right| \leq \mu \tag{3}$$

In the microscope interpretation, if $(x, u) \in \rho_{\varepsilon,b}$ means that x and u represent the same "real" point in the specimen.

Therefore a metric profile gives two types of information:

- a distance estimate like (3) from point 4,
- an "approximate shape" estimate, like in the points 1–3, where we see that two sets, namely the balls $\bar{B}_{d_b}(x_b, \varepsilon)$ and $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$, are approximately isometric.

The simplest metric profile is one with $(\bar{B}(x_b, 1), d_b, x_b) = (X, d_b, x)$. In this case we see that $\rho_{\varepsilon,b}$ is approximately an ε dilatation with base point x.

This observation leads us to a particular class of (pointed) metric spaces, namely the metric cones.

Definition 11. *A metric cone (X, d, x) is a locally compact metric space (X, d), with a marked point $x \in X$ such that for any $a, b \in (0, 1]$ we have*

$$\mathbb{P}^m(a, x) = \mathbb{P}^m(b, x)$$

Metric cones have dilatations. By this we mean the following

Definition 12. *Let (X, d, x) be a metric cone. For any $\varepsilon \in (0, 1]$ a dilatation is a function $\delta_\varepsilon^x : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)$ such that*

- $\delta_\varepsilon^x(x) = x$,
- *for any $u, v \in X$ we have*

$$d\left(\delta_\varepsilon^x(u), \delta_\varepsilon^x(v)\right) = \varepsilon \, d(u, v)$$

The existence of dilatations for metric cones comes from the definition 11. Indeed, dilatations are just isometries from $(\bar{B}(x, 1), d, x)$ to $(\bar{B}, \frac{1}{a} d, x)$.

Metric cones are good candidates for being tangent spaces in the metric sense.

Definition 13. *A (locally compact) metric space (M, d) admits a (metric) tangent space in $x \in M$ if the associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ (as in definition 9) admits a prolongation by continuity in $\varepsilon = 0$, i.e if the following limit exists:*

$$[T_x M, d^x, x] = \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \tag{4}$$

The connection between metric cones, tangent spaces and metric profiles in the abstract sense is made by the following proposition.

Proposition 14. *The associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ of a metric space (M, d) for a fixed $x \in M$ is a metric profile in the sense of the definition 10 if and only if the space (M, d) admits a tangent space in x. In such a case the tangent space is a metric cone.*

Proof. A tangent space $[V, d_v, v]$ exists if and only if we have the limit from the relation (4). In this case there exists a prolongation by continuity to $\varepsilon = 0$ of the metric profile $\mathbb{P}^m(\cdot, x)$. The prolongation is a metric profile in the sense of definition 10. Indeed, we have still to check the property (b). But this is trivial, because for any $\varepsilon, b > 0$, sufficiently small, we have

$$\mathbb{P}^m(\varepsilon b, x) = \mathbb{P}^m_{d_b}(\varepsilon, x)$$

where $d_b = (1/b)d$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x]$.

Finally, let us prove that the tangent space is a metric cone. For any $a \in (0, 1]$ we have

$$\left[\bar{B}(x, 1), \frac{1}{a} d^x, x\right] = \lim_{\varepsilon \to 0} \mathbb{P}^m(a\varepsilon, x)$$

Therefore

$$\left[\bar{B}(x, 1), \frac{1}{a} d^x, x\right] = [T_x M, d^x, x]$$

□

2.4. **Length in metric spaces.** For a detailed introduction into the subject see for example [1], chapter 1.

Definition 15. *The* **(upper) dilatation of a map** $f : X \to Y$ *between metric spaces, in a point* $u \in Y$ *is*

$$Lip(f)(u) = \limsup_{\varepsilon \to 0} \ \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v, w)} \ : \ v \neq w \ , \ v, w \in B(u, \varepsilon) \right\}$$

In the particular case of a derivable function $f : \mathbb{R} \to \mathbb{R}^n$ the upper dilatation is $Lip(f)(t) = \|\dot{f}(t)\|$.

A function $f : (X, d) \to (Y, d')$ is Lipschitz if there is a positive constant C such that for any $x, y \in X$ we have $d'(f(x), f(y)) \leq C\, d(x, y)$. The number $Lip(f)$ is the smallest such positive constant. Then for any $x \in X$ we have the obvious relation $Lip(f)(x) \ \leq \ Lip(f)$.

A curve is a continuous function $c : [a, b] \to X$. The image of a curve is called path. Length measures paths. Therefore length does not depends on the reparameterization of the path and it is additive with respect to concatenation of paths.

Definition 16. *In a metric space (X, d) there are several ways to define the length:*

(a) *The* **length of a curve with L^1 upper dilatation** $c : [a, b] \to X$ *is*

$$L(f) = \int_a^b Lip(c)(t) \ dt$$

(b) *The* **variation of a curve** $c : [a,b] \rightarrow X$ *is the quantity* $Var(c) =$

$$= \sup \left\{ \sum_{i=0}^{n} d(c(t_i), c(t_{i+1})) \quad : a = t_0 < t_1 < ... < t_n < t_{n+1} = b \right\}$$

(c) *The* **length of the path** $A = c([a,b])$ *is the one-dimensional Hausdorff measure of the path.:*

$$l(A) \;=\; \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i \in I} diam\ E_i \ : \ diam\ E_i < \delta \ , \quad A \subset \bigcup_{i \in I} E_i \right\}$$

The definitions are not equivalent. For Lipschitz curves the first two definitions agree. For simple Lipschitz curves all definitions agree.

Theorem 17. *For each Lipschitz curve $c : [a,b] \rightarrow X$, we have $L(c) \;=\; Var(c) \;\geq\; \mathcal{H}^1(c([a,b]))$.*
If c is moreover injective then $\mathcal{H}^1(c([a,b])) \;=\; Var(f)$.

An important tool used in the proof of the previous theorem is the geometrically obvious, but not straightforward to prove in this generality, Reparametrisation Theorem.

Theorem 18. *Any Lipschitz curve admits a reparametrisation $c : [a,b] \rightarrow A$ such that $Lip(c)(t) = 1$ for almost any $t \in [a,b]$.*

Definition 19. *We shall denote by l_d the* **length functional induced by the distance** *d, defined only on the family of Lipschitz curves. If the metric space (X,d) is connected by Lipschitz curves, then the length induces a new distance d_l, given by:*

$$d_l(x,y) \;=\; \inf \; \{ l_d(c([a,b])) \ : \ c : [a,b] \rightarrow X \quad \text{Lipschitz} \ ,$$

$$c(a) = x \ , \ c(b) = y \}$$

A **length metric space** *is a metric space (X,d), connected by Lipschitz curves, such that $d = d_l$.*

From theorem 17 we deduce that Lipschitz curves in complete length metric spaces are absolutely continuous. Indeed, here is the definition of an absolutely continuous curve (definition 1.1.1, chapter 1, [1]).

Definition 20. *Let (X,d) be a complete metric space. A curve $c : (a,b) \rightarrow X$ is* **absolutely continuous** *if there exists $m \in L^1((a,b))$ such that for any $a < s \leq t < b$ we have*

$$d(c(s), c(t)) \leq \int_s^t m(r) \ dr.$$

Such a function m is called a **upper gradient** *of the curve c.*

According to theorem 17, for a Lipschitz curve $c : [a,b] \to X$ in a complete length metric space such a function $m \in L^1((a,b))$ is the upper dilatation $Lip(c)$. More can be said about the expression of the upper dilatation. We need first to introduce the notion of metric derivative of a Lipschitz curve.

Definition 21. *A curve $c : (a,b) \to X$ is* **metrically derivable** *in $t \in (a,b)$ if the limit*

$$md(c)(t) = \lim_{s \to t} \frac{d(c(s), c(t))}{\mid s - t \mid}$$

exists and it is finite. In this case $md(c)(t)$ is called the **metric derivative** *of c in t.*

For the proof of the following theorem see [1], theorem 1.1.2, chapter 1.

Theorem 22. *Let (X,d) be a complete metric space and $c : (a,b) \to X$ be an absolutely continuous curve. Then c is metrically derivable for $\mathcal{L}^1$-a.e. $t \in (a,b)$. Moreover the function $md(c)$ belongs to $L^1((a,b))$ and it is minimal in the following sense: $md(c)(t) \leq m(t)$ for $\mathcal{L}^1$-a.e. $t \in (a,b)$, for each upper gradient m of the curve c.*

3. Metric spaces with dilations

We shall use here a slightly particular version of dilatation structures. For the general definition of a dilatation structure see [4] (the general definition applies for dilatation structures over ultrametric spaces as well).

Definition 23. *Let (X,d) be a complete metric space such that for any $x \in X$ the closed ball $\bar{B}(x,3)$ is compact. A* **dilatation structure** *(X,d,δ) over (X,d) is the assignment to any $x \in X$ and $\varepsilon \in (0,+\infty)$ of a invertible homeomorphism, defined as: if $\varepsilon \in (0,1]$ then $\delta_\varepsilon^x : U(x) \to V_\varepsilon(x)$, else $\delta_\varepsilon^x : W_\varepsilon(x) \to U(x)$, such that the following axioms are satisfied:*

A0. *there are numbers $1 < A < B$ such that for any $x \in X$ and any $\varepsilon \in (0,1)$ we have the following string of inclusions:*

$$B_d(x,\varepsilon) \subset \delta_\varepsilon^x B_d(x,A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x,B)$$

Moreover for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u, v \in \bar{B}_d(x,R)$ and all $\varepsilon \in (0,\varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) \ .$$

A1. *We have $\delta_\varepsilon^x x = x$ for any point x. We also have $\delta_1^x = id$ for any $x \in X$. Let us define the topological space*

$$dom\,\delta = \{(\varepsilon,x,y) \in (0,+\infty) \times X \times X \ : \ if\ \varepsilon \leq 1 \ then\ y \in U(x) \ ,$$
$$else\ y \in W_\varepsilon(x)\}$$

with the topology inherited from $(0,+\infty) \times X \times X$ endowed with the product topology. Consider also $Cl(dom\,\delta)$, the closure of $dom\,\delta$ in $[0,+\infty) \times X \times X$.

The function $\delta : dom\,\delta \rightarrow X$ defined by $\delta(\varepsilon, x, y) = \delta_{\varepsilon}^{x} y$ is continuous. Moreover, it can be continuously extended to the set $Cl(dom\,\delta)$ and we have

$$\lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon}^{x} y \,=\, x$$

A2. *For any $x, \in X$, $\varepsilon, \mu \in (0, +\infty)$ and $u \in U(x)$ we have the equality:*

$$\delta_{\varepsilon}^{x} \delta_{\mu}^{x} u = \delta_{\varepsilon\mu}^{x} u$$

whenever one of the sides are well defined.

A3. *For any x there is a distance function $(u, v) \mapsto d^{x}(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that*

$$\lim_{\varepsilon \rightarrow 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v) \;-\; d^{x}(u, v) \mid \;:\, u, v \in \bar{B}_{d}(x, A) \right\} \;=\; 0$$

uniformly with respect to x in compact set.

The **dilatation structure is strong** *if it satisfies the following supplementary condition:*

A4. *Let us define $\Delta_{\varepsilon}^{x}(u, v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^{x} u} \delta_{\varepsilon}^{x} v$. Then we have the limit*

$$\lim_{\varepsilon \rightarrow 0} \Delta_{\varepsilon}^{x}(u, v) = \Delta^{x}(u, v)$$

uniformly with respect to x, u, v in compact set.

We shall use many times from now the words "sufficiently close". This deserves a definition.

Definition 24. *Let (X, d, δ) be a strong dilatation structure. We say that a property $\mathcal{P}(x_1, x_2, x_3, ...)$ holds for $x_1, x_2, x_3, ...$* **sufficiently close** *if for any compact, non empty set $K \subset X$, there is a positive constant $C(K) > 0$ such that $\mathcal{P}(x_1, x_2, x_3, ...)$ is true for any $x_1, x_2, x_3, ... \in K$ with $d(x_i, x_j) \leq C(K)$.*

We shall look at dilatation structures from the metric point of view, by using Gromov-Hausdorff distance and metric profiles.

We state the interpretation of the Axiom A3 as a theorem. But before a definition: we denote by (δ, ε) the distance on

$$\bar{B}_{d^{x}}(x, 1) = \{y \in X \colon\, d^{x}(x, y) \leq 1\}$$

given by

$$(\delta, \varepsilon)(u, v) = \frac{1}{\varepsilon} d(\delta_{\varepsilon}^{x} u, \delta_{\varepsilon}^{x} v)$$

Theorem 25. *Let (X, d, δ) be a dilatation structure. The following are consequences of the Axioms A0 - A3 only:*

(a) *for all $u, v \in X$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$ and all $\mu \in (0, A)$ we have*

$$d^{x}(u, v) = \frac{1}{\mu} d^{x}(\delta_{\mu}^{x} u, \delta_{\mu}^{x} v)$$

We shall say that d^{x} has the cone property with respect to dilatations.

(b) *The curve* $\varepsilon > 0 \mapsto \mathbb{P}^x(\varepsilon) = [\bar{B}_{d^x}(x,1), (\delta, \varepsilon), x]$ *is a metric profile.*

Proof. (a) For $\varepsilon, \mu \in (0,1)$ we have

$$\left| \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu u, \delta^x_\varepsilon \delta^x_\mu v) - d^x(u,v) \right| \leq \left| \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_\varepsilon \delta^x_\mu u) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} v, \delta^x_\varepsilon \delta^x_\mu v) \right| + \left| \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_{\varepsilon\mu} v) - d^x(u,v) \right|$$

Use now the Axioms A2 and A3 and pass to the limit with $\varepsilon \to 0$. This gives the desired equality.

(b)We have to prove that $\mathbb{P}^x$ is a metric profile. For this we have to compare two pointed metric spaces:

$$\left(\bar{B}_{d^x}(x,1), (\delta^x, \varepsilon\mu), x\right) \quad \text{and} \quad \left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1), \frac{1}{\mu}(\delta^x, \varepsilon), x\right)$$

Let $u \in X$ such that

$$\frac{1}{\mu}(\delta^x, \varepsilon)(x,u) \leq 1$$

This means that

$$\frac{1}{\varepsilon} d(\delta^x_\varepsilon x, \delta^x_\varepsilon u) \ \leq \ \mu$$

Further use the Axioms A1, A2 and the cone property proved before:

$$\frac{1}{\varepsilon} d^x(\delta^x_\varepsilon x, \delta^x_\varepsilon u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

therefore,

$$d^x(x,u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

It follows that for any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1)$ we can choose $w(u) \in \bar{B}_{d^x}(x,1)$ such that

$$\frac{1}{\mu} d^x(u, \delta^x_\mu w(u)) = \mathcal{O}(\varepsilon)$$

We want to prove that

$$\mid \frac{1}{\mu}(\delta^x, \varepsilon)(u_1, u_2) - (\delta^x, \varepsilon\mu)(w(u_1), w(u_2)) \mid \leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)$$

This goes as follows:

$$\begin{aligned} &|\frac{1}{\mu}(\delta^x, \varepsilon)(u_1, u_2) - (\delta^x, \varepsilon\mu)(w(u_1), w(u_2))| \ = \\ &= \left| \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} w(u_1), \delta^x_{\varepsilon\mu} w(u_2)) \right| \\ &\leq \mathcal{O}(\varepsilon\mu) + \left| \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon u_1, \delta^x_\varepsilon u_2) - \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu w(u_1), \delta^x_\varepsilon \delta^x_\mu w(u_2)) \right| \\ &\leq \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \mid d^x(u_1, u_2) - d^x(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \mid \end{aligned}$$

In order to obtain the last estimate we used twice the Axiom A3. We proceed as follows:

$$\mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \mid d^{x}(u_1, u_2) - d^{x}(\delta^x_\mu w(u_1), \delta^x_\mu w(u_2)) \mid \leq$$

$$\leq \mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} d^{x}(u_1, \delta^x_\mu w(u_1)) \ + \ \frac{1}{\mu} d^{x}(u_1, \delta^x_\mu w(u_2))$$

$$\leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon)$$

This shows that the property (b) of a metric profile is satisfied. The property (a) is proved in the Theorem 26. $\square$

The following theorem is related to Mitchell [17] Theorem 1, concerning sub-riemannian geometry.

Theorem 26. *In the hypothesis of theorem 25, we have the following limit:*

$$\lim_{\varepsilon \rightarrow 0} \ \frac{1}{\varepsilon} \sup \left\{ \mid d(u,v) - d^{x}(u,v) \mid : d(x,u) \leq \varepsilon, \ d(x,v) \leq \varepsilon \right\} = 0$$

Therefore if d^x is a true (i.e. nondegenerate) distance, then (X, d) admits a metric tangent space in x.

Moreover, the metric profile $[\bar{B}_{d^x}(x,1), (\delta, \varepsilon), x]$ is almost nice, in the following sense. Let $c \in (0,1)$. Then we have the inclusion

$$\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x, c) \right) \ \subset \ \bar{B}_{d^x}(x,1)$$

Moreover, the following Gromov-Hausdorff distance is of order $\mathcal{O}(\varepsilon)$ for μ fixed (that is the modulus of convergence $\mathcal{O}(\varepsilon)$ does not depend on μ):

$$\mu \ d_{GH} \left([\bar{B}_{d^x}(x,1), (\delta^x, \varepsilon), x], \ [\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x, c) \right), (\delta^x, \varepsilon\mu), x] \right) = \mathcal{O}(\varepsilon)$$

For another Gromov-Hausdorff distance we have the estimate

$$d_{GH} \left([\bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x, c), \frac{1}{\mu}(\delta^x, \varepsilon), x] \ , \ [\delta^x_{\mu^{-1}} \left(\bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x, c) \right), (\delta^x, \varepsilon\mu), x] \right) = \mathcal{O}(\varepsilon\mu)$$

when $\varepsilon \in (0, \varepsilon(c))$.

Proof. We start from the Axioms A0, A3 and we use the cone property. By A0, for $\varepsilon \in (0,1)$ and $u, v \in \bar{B}_d(x, \varepsilon)$ there exist $U, V \in \bar{B}_d(x, A)$ such that

$$u = \delta^x_\varepsilon U, v = \delta^x_\varepsilon V.$$

By the cone property we have

$$\frac{1}{\varepsilon} \mid d(u,v) - d^x(u,v) \mid = \left| \frac{1}{\varepsilon} d(\delta^x_\varepsilon U, \delta^x_\varepsilon V) - d^x(U,V) \right|$$

By A2 we have

$$\left| \frac{1}{\varepsilon} d(\delta^x_\varepsilon U, \delta^x_\varepsilon V) - d^x(U,V) \right| \ \leq \ \mathcal{O}(\varepsilon)$$

This proves the first part of the theorem.

For the second part of the theorem take any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)$. Then we have

$$d^x(x,u) \leq c\mu + \mathcal{O}(\varepsilon)$$

Then there exists $\varepsilon(c) > 0$ such that for any $\varepsilon \in (0, \varepsilon(c))$ and u in the mentioned ball we have

$$d^x(x,u) \leq \mu$$

In this case we can take directly $w(u) = \delta^x_{\mu^{-1}} u$ and simplify the string of inequalities from the proof of Theorem 25, point (b), to get eventually the three points from the second part of the theorem. □

4. Length metric spaces with dilations

Consider (X,d) a complete, locally compact metric space, and a triple (X,d,δ) which satisfies A0, A1, A2. Denote by $Lip([0,1],X,d)$ the space of d-Lipschitz curves $c : [0,1] \to X$. Let also l_d denote the length functional associated to the distance d.

4.1. Gamma-convergence of length functionals.

Definition 27. *For any $\varepsilon \in (0,1)$ we define the* **length functional**

$$l_\varepsilon : \mathcal{L}_\varepsilon(X,d,\delta) \to [0,+\infty] \quad , \quad l_\varepsilon(x,c) = l^x_\varepsilon(c) = \frac{1}{\varepsilon} l_d(\delta^x_\varepsilon c)$$

The domain of definition of the functional l_ε is the space:

$$\mathcal{L}_\varepsilon(X,d,\delta) = \{(x,c) \in X \times \mathcal{C}([0,1],X) \, : \, c : [0,1] \in U(x) \, ,$$
$$\delta^x_\varepsilon c \ is \ d - Lip \ and \ Lip(\delta^x_\varepsilon c) \leq 2\, l_d(\delta^x_\varepsilon c)\}$$

The last condition from the definition of $\mathcal{L}_\varepsilon(X,d,\delta)$ is a selection of parameterization of the path $c([0,1])$. Indeed, by the reparameterization theorem, if $\delta^x_\varepsilon c : [0,1] \to (X,d)$ is a d-Lipschitz curve of length $L = l_d(\delta^x_\varepsilon c)$ then $\delta^x_\varepsilon c([0,1])$ can be reparameterized by length, that is there exists a increasing function $\phi : [0,L] \to [0,1]$ such that $c' = \delta^x_\varepsilon c \circ \phi$ is a d-Lipschitz curve with $Lip(c') \leq 1$. But we can use a second affine reparameterization which sends $[0,L]$ back to $[0,1]$ and we get a Lipschitz curve $c"$ with $c"([0,1]) = c'([0,1])$ and $Lip(c") \leq 2 l_d(c)$.

We shall use the following definition of Gamma-convergence (see the book [14] for the notion of Gamma-convergence). Notice the use of convergence of sequences only in the second part of the definition.

Definition 28. *Let Z be a metric space with distance function D and $(l_\varepsilon)_{\varepsilon>0}$ be a family of functionals $l_\varepsilon : Z_\varepsilon \subset Z \to [0,+\infty]$. Then l_ε* **Gamma-converges** *to the functional $l : Z_0 \subset Z \to [0,+\infty]$ if:*

(a) *(**liminf inequality**) for any function $\varepsilon \in (0,\infty) \mapsto x_\varepsilon \in Z_\varepsilon$ such that $\lim_{\varepsilon\to 0} x_\varepsilon = x_0 \in Z_0$ we have*

$$l(x_0) \leq \liminf_{\varepsilon\to 0} l_\varepsilon(x_\varepsilon)$$

(b) *(**existence of a recovery sequence**) For any $x_0 \in Z_0$ and for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in Z_{\varepsilon_n}$ for any $n \in \mathbb{N}$, such that*

$$l(x_0) = \lim_{n\to\infty} l_{\varepsilon_n}(x_n)$$

We shall take as the metric space Z the space $X \times \mathcal{C}([0,1], X)$ with the distance

$$D((x,c),(x',c')) = \max\{d(x,x')\,,\, \sup\{d(c(t),c'(t))\,:\, t\in[0,1]\}\}$$

Let $\mathcal{L}(X,d,\delta)$be the class of all $(x,c) \in X \times \mathcal{C}([0,1],X)$ which appear as limits $(x_n,c_n) \to (x,c)$, with $(x_n,c_n) \in \mathcal{L}_{\varepsilon_n}(X,d,\delta)$, the family $(c_n)_n$ is d-equicontinuous and $\varepsilon_n \to 0$ as $n \to \infty$.

Definition 29. *A triple (X,d,δ) is a* **length dilatation structure** *if (X,d) is a complete, locally compact metric space such that A0, A1, A2, are satisfied, together with the following axioms:*

A3L. *there is a functional $l : \mathcal{L}(X,d,\delta) \to [0,+\infty]$ such that for any $\varepsilon_n \to 0$ as $n \to \infty$ the sequence of functionals l_{ε_n} Gamma-converges to the functional l.*

A4+ *Let us define $\Delta^x_\varepsilon(u,v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v$ and $\Sigma^x_\varepsilon(u,v) = \delta^x_{\varepsilon^{-1}} \delta^{\delta^x_\varepsilon u}_\varepsilon v$. Then we have the limits*

$$\lim_{\varepsilon\to 0} \Delta^x_\varepsilon(u,v) = \Delta^x(u,v)$$
$$\lim_{\varepsilon\to 0} \Sigma^x_\varepsilon(u,v) = \Sigma^x(u,v)$$

uniformly with respect to x,u,v in compact set.

Remark 1. For strong dilatation structures the axioms A0 - A4 imply A4+. The transformations $\Sigma^x_\varepsilon(u,\cdot)$ have the interpretation of approximate left translations in the tangent space of (X,d) at x.

For any $\varepsilon \in (0,1)$ and any $x \in X$ the length functional l^x_ε induces a distance on $U(x)$:

$$\mathring{d}^x_\varepsilon(u,v) = \inf\{l^x_\varepsilon(c)\,:\,(x,c)\in\mathcal{L}_\varepsilon(X,d,\delta)\,,\, c(0)=u\,,\, c(1)=v\}$$

In the same way the length functional l from A3L induces a distance $\mathring{d}^x$ on $U(x)$.

Gamma-convergence implies that

$$\mathring{d}^x(u,v) \geq \limsup_{\varepsilon\to 0} \mathring{d}^x_\varepsilon(u,v) \tag{5}$$

Remark 2. Without supplementary hypotheses we cannot prove A3 from A3L, that is in principle length dilatation structures are not strong dilatation structures.

5. The Radon-Nikodym property

5.1. **Differentiability with respect to dilatation structures.** For any strong dilatation structure or length dilatation structure there is an associated notion of differentiability (section 7.2 [4]). First we need the definition of a morphism of conical groups.

Definition 30. *Let (N, δ) and $(M, \bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.*

The definition of the derivative, or differential, with respect to dilatations structures follows. In the case of a pair of Carnot groups this is just the definition of the Pansu derivative introduced in [18].

Definition 31. *Let (X, d, δ) and $(Y, \overline{d}, \overline{\delta})$ be two strong dilatation structures or length and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $D\, f(x) : T_x X \to T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that*

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(f\left(\delta^x_\varepsilon u\right), \overline{\delta}^{f(x)}_\varepsilon D\, f(x)(u) \right) \ : \ d(x, u) \leq \varepsilon \right\} = 0, \tag{6}$$

The morphism $D\, f(x)$ is called the derivative, or differential, of f at x.

The definition also makes sense if the function f is defined on a open subset of (X, d).

5.2. **The Radon-Nikodym property.**

Definition 32. *A strong dilatation structure or a length dilatation structure has the* **Radon-Nikodym property (or rectifiability property, or RNP)** *if any Lipschitz curve $c : [a, b] \to (X, d)$ is derivable almost everywhere.*

5.3. **Two examples.** The following two easy examples will show that not any strong dilatation structure has the Radon-Nikodym property.

For $(X, d) = (\mathbb{V}, d)$, a real, finite dimensional, normed vector space, with distance d induced by the norm, the (usual) dilatations δ^x_ε are given by:

$$\delta^x_\varepsilon y \ = \ x + \varepsilon(y - x)$$

Dilatations are defined everywhere.

There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ = \ d(u, v) \ ,$$

therefore for any $x \in X$ we have $d^x = d$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v \ = \ x + \varepsilon(u - x) + \frac{1}{\varepsilon} \left(x + \varepsilon(v - x) - x - \varepsilon(u - x) \right) \ =$$

$$= \ x + \varepsilon(u-x) + v - u$$

therefore this quantity converges to

$$x + v - u \ = \ x + (v-x) - (u-x)$$

as $\varepsilon \to 0$. The axiom A4 is verified.

This dilatation structure has the Radon-Nikodym property.

Further is an example of a dilatation structure which does not have the Radon-Nikodym property. Take $X = \mathbb{R}^2$ with the euclidean distance d. For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilatations

$$\delta_\varepsilon x = \varepsilon^z x \ .$$

It is easy to check that $(\mathbb{R}^2, d, \delta)$ is a dilatation structure, with dilatations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y-x)$$

Two such dilatation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other interesting properties of these dilatation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. It means that such dilatation structure does not have the Radon-Nikodym property.

The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a $\mathcal{C}^\infty$ function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is not holomorphic).

5.4. **Length formula from Radon-Nikodym property.**

Definition 33. *In a normed conical group N we shall denote by $D(N)$ the set of all $u \in N$ with the property that $\varepsilon \in ((0,\infty),+) \mapsto \delta_\varepsilon u \in N$ is a morphism of groups.*

$D(N)$ is always non empty, because it contains the neutral element of N. $D(N)$ is also a cone, with dilatations δ_ε, and a closed set.

Proposition 34. *Let (X, d, δ) be a strong dilatation structure. Then the following are equivalent:*

(a) *(X, d, δ) has the Radon-Nikodym property;*
(b) *for any Lipschitz curve $c : [a,b] \to (X,d)$, for almost every $t \in [a,b]$ there is $\dot{c}(t) \in D(T_{c(t}(X,d,\delta))$ such that*

$$\frac{1}{\varepsilon} d(c(t+\varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0$$

$$\frac{1}{\varepsilon} d(c(t-\varepsilon), \delta_\varepsilon^{c(t)} inv^{c(t)}(\dot{c}(t))) \to 0$$

Proof. It is straightforward that a conical group morphism $f : \mathbb{R} \to N$ is defined by its value $f(1) \in N$. Indeed, for any $a > 0$ we have $f(a) = \delta_a f(1)$ and for any $a < 0$ we have $f(a) = \delta_a f(1)^{-1}$. From the morphism property we also deduce that

$$\delta v = \left\{ \delta_a v \, : \, a > 0, v = f(1) \text{ or } v = f(1)^{-1} \right\}$$

is a one parameter group and that for all $\alpha, \beta > 0$ we have $\delta_{\alpha+\beta} u = \delta_\alpha u \, \delta_\beta u$. We have therefore a bijection between conical group morphisms $f : \mathbb{R} \to (N, \delta)$ and elements of $D(N)$.

A Lipschitz curve $c : [a, b] \to (X, d)$ is derivable in $t \in (a, b)$ if and only if there is a morphism of normed conical groups $f : \mathbb{R} \to T_{c(t}(X, d, \delta)$ such that for any $a \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \, d(c(t + \varepsilon a), \delta_\varepsilon^{c(t)} f(a)) \ = \ 0$$

Take $\dot{c}(t) = f(1)$. Then $\dot{c}(t) \in D(T_{c(t}(X, d, \delta))$. For any $a > 0$ we have $f(a) = \delta_a^{c(t)} \dot{c}(t)$; otherwise if $a < 0$ we have $f(a) = \delta_a^{c(t)} \, inv^{c(t)} \, \dot{c}(t)$. This implies the equivalence stated on the proposition. $\square$

Theorem 35. *Let (X, d, δ) be a strong dilatation structure with the Radon-Nikodym property, over a complete length metric space (X, d). Then for any $x, y \in X$ we have*

$$d(x, y) \ = \ \inf \left\{ \int_a^b d^{c(t)}(c(t), \dot{c}(t)) \ dt \ : \ c : [a, b] \to X \ \mathit{Lipschitz} \ , \right.$$
$$\left. c(a) = x, c(b) = y \right\}$$

Proof. From theorem 22 we deduce that for almost every $t \in (a, b)$ the upper dilatation of c in t can be expressed as:

$$Lip(c)(t) = \lim_{s \to t} \frac{d(c(s), c(t))}{\mid s - t \mid}$$

If the dilatation structure has the Radon-Nikodym property then for almost every $t \in [a, b]$ there is $\dot{c}(t) \in D(T_{c(t)} X)$ such that

$$\frac{1}{\varepsilon} d(c(t + \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \to 0$$

Therefore for almost every $t \in [a, b]$ we have

$$Lip(c)(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(c(t + \varepsilon), c(t)) = d^{c(t)}(c(t), \dot{c}(t))$$

The formula for length follows from here. $\square$

A straightforward consequence is that the distance d is uniquely determined by the "distribution" $x \in X \mapsto D(T_x(X, d, \delta))$ and the function which associates to any $x \in X$ the "norm" $\| \cdot \|_x : D(T_x(X, d, \delta)) \to [0, +\infty)$.

Corollary 36. *Let (X, d, δ) and $(X, \bar{d}, \bar{\delta})$ be two strong dilatation structures with the Radon-Nikodym property , which are also complete length metric spaces, such that for any $x \in X$ we have $D(T_x(X, d, \delta)) = D(T_x(X, d', \delta'))$ and $d^x(x, u) = \bar{d}^x(x, u)$ for any $u \in D(T_x(X, d, \delta))$. Then $d = d'$.*

5.5. **Equivalent dilatation structures and their distributions.**

Definition 37. Two strong dilatation structures (X, δ, d) *and* $(X, \overline{\delta}, \overline{d})$ **are equivalent** *if*

(a) *the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz and*
(b) *for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overline{d} \left(\delta^x_\varepsilon u, \overline{\delta}^x_\varepsilon Q^x(u) \right) = 0, \tag{7}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d \left(\overline{\delta}^x_\varepsilon u, \delta^x_\varepsilon P^x(u) \right) = 0, \tag{8}$$

uniformly with respect to x, u in compact sets.

Proposition 38. *(X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if and only if*

(a) *the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz,*
(b) *for any $x \in X$ there are conical group morphisms:*

$$P^x : T_x(X, \overline{\delta}, \overline{d}) \to T_x(X, \delta, d) \text{ and } Q^x : T_x(X, \delta, d) \to T_x(X, \overline{\delta}, \overline{d})$$

such that the following limits exist

$$\lim_{\varepsilon \to 0} \left(\overline{\delta}^x_\varepsilon \right)^{-1} \delta^x_\varepsilon(u) = Q^x(u), \tag{9}$$

$$\lim_{\varepsilon \to 0} \left(\delta^x_\varepsilon \right)^{-1} \overline{\delta}^x_\varepsilon(u) = P^x(u), \tag{10}$$

and are uniform with respect to x, u in compact sets.

The next theorem shows a link between the tangent bundles of equivalent dilatation structures.

Theorem 39. *Let (X, d, δ) and $(X, \overline{d}, \overline{\delta})$ be equivalent strong dilatation structures. Then for any $x \in X$ and any $u, v \in X$ sufficiently close to x we have:*

$$\overline{\Sigma}^x(u, v) = Q^x \left(\Sigma^x \left(P^x(u), P^x(v) \right) \right). \tag{11}$$

The two tangent bundles are therefore isomorphic in a natural sense.

As a consequence, the following corollary is straightforward.

Corollary 40. *Let* (X, d, δ) *and* $(X, \overline{d}, \overline{\delta})$ *be equivalent strong dilatation structures. Then for any* $x \in X$ *we have*

$$Q^x(D(T_x(X, \delta, d))) \; = \; D(T_x(X, \overline{\delta}, \overline{d}))$$

If (X, d, δ) *has the Radon-Nikodym property , then* $(X, \overline{d}, \overline{\delta})$ *has the same property.*

Suppose that (X, d, δ) *and* $(X, \overline{d}, \overline{\delta})$ *are complete length spaces with the Radon-Nikodym property . If the functions* P^x, Q^x *from definition 37 (b) are isometries, then* $d = \overline{d}$*.*

6. Tempered dilatation structures

The notion of a tempered dilatation structure is inspired by the results from Venturini [21] and Buttazzo, De Pascale and Fragalà [13].

The examples of length dilatation structures from this section are provided by the extension of some results from [13] (propositions 2.3, 2.6 and a part of theorem 3.1) to dilatation structures.

We recall some definition2.1 from [13] section 2. Let Ω be a given connected open subset of $\mathbb{R}^N$ endowed with the distance induced by the euclidean norm. Given two positive constants $c < C$, let $D(\Omega)$ be the class of all length distances on Ω such that

$$c\|u - v\| \; \leq \; d(u, v) \; \leq \; C\|u - v\| \tag{12}$$

for all $u, v \in \Omega$. We suppose that $D(\Omega$ is not empty. $D(\Omega)$ is endowed with the topology of uniform convergence on compact subsets of $\Omega \times \Omega$.

To any $d \in D(\Omega)$ is associated the function

$$\phi_d(x, u) \; = \; \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \, d(x, x + \varepsilon u)$$

This function is measurable in x and convex positively one-homogeneous in z. From (12) we see that ϕ_d has the property:

$$c\|z\| \; \leq \; \phi_d(x, z) \; \leq \; C\|z\| \tag{13}$$

By proposition 2.4 [13] the function ϕ_d allows to write in an integral form he length functional l_d associated to d: for any lipschitz curve $c : [0, 1] \to \Omega$ we have

$$l_d(c) \; = \; \int_0^1 \phi_d(c(t), \dot{c}(t)) \; \mathrm{d}t$$

With the first example of a dilatation structure with the Radon-Nikodym in mind (see subsection 5.3), we can easily rewrite this in terms of dilatation structures. Indeed, it suffices to replace $\mathbb{R}^N$ with the euclidean distance by a metric space $(X, \bar{d})$ endowed with a dilatation structure. Then we may choose to see the $\|z\|$

from relation (13) as the distance $\bar{d}^x(u,z)$. Finally, instead of (12), (13), we may write $d \in mathcal D(\Omega, \bar{d}, \bar{\delta})$ if

$$c\, \bar{d}^x(u,v) \, \leq \, \frac{1}{\varepsilon}\, d(\bar{\delta}^x_\varepsilon u, \bar{\delta}^x_\varepsilon v) \, \leq \, C\, \bar{d}^x(u,v)$$

and ϕ_d could also be rewritten as:

$$\phi_d(x,u) \, = \, \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\, d(x, \delta^x_\varepsilon u)$$

The euclidean distance, or the distance $\bar{d}$ is here fixed, and the class $D(\Omega)$ is defined relatively to $\bar{d}$. Remark that for $\bar{d}$ being the euclidean distance in $X = \mathbb{R}^N$, it is true that $\bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. This inspired us to call such dilatation structures "tempered".

The construction is presented further in detail. The following definition gives a class of distances $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$, associated to a strong dilatation structure $(\Omega, \bar{d}, \bar{\delta})$, which in some sense generalizes the class of distances $\mathcal{D}(\Omega)$ from [13], definition 2.1.

Definition 41. *For any strong dilatation structure $(\Omega, \bar{d}, \bar{\delta})$ and constants $0 < c < C$ we define the class $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ of all distance functions d on Ω such that*

(a) *d is a length distance,*

(b) *for any $\varepsilon > 0$ and any x, u, v sufficiently closed we have:*

$$c\, \bar{d}^x(u,v) \, \leq \, \frac{1}{\varepsilon}\, d(\bar{\delta}^x_\varepsilon u, \bar{\delta}^x_\varepsilon v) \, \leq \, C\, \bar{d}^x(u,v) \tag{14}$$

The dilatation structure $(\Omega, \bar{d}, \bar{\delta})$ is **tempered** *if there are constants c, C such that $\bar{d} \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$.*

On $\mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ we put the topology of uniform convergence (induced by distance $\bar{d}$) on compact subsets of $\Omega \times \Omega$.

To any distance $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ we associate the function:

$$\phi_d(x,u) \, = \, \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\, d(x, \delta^x_\varepsilon u)$$

defined for any $x, u \in \Omega$ sufficiently close. We have therefore

$$c\, \bar{d}^x(x,u) \, \leq \, \phi_d(x,u) \, \leq \, C\, \bar{d}^x(x,u) \tag{15}$$

Notice that if $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then for any x, u, v sufficiently close we have

$$-\bar{d}(x,u)\, O(\bar{d}(x,u)) \, + \, c\, \bar{d}^x(u,v) \, \leq$$

$$\leq \, d(u,v) \, \leq \, C\, \bar{d}^x(u,v) \, + \, \bar{d}(x,u)\, O(\bar{d}(x,u))$$

If $c : [0,1] \rightarrow \Omega$ is a d-Lipschitz curve and $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ then we may decompose it in a finite family of curves $c_1, ..., c_n$ (with n depending on c) such that there are $x_1, ..., x_n \in \Omega$ with c_k is $\bar{d}^{x_k}$-Lipschitz. Indeed, the image of the curve $c([0,1])$ is compact, therefore we may cover it with a finite number of balls $B(c(t_k), \rho_k, \bar{d}^{c(t_k)})$ and apply (14). If moreover $(\Omega, \bar{d}, \bar{\delta})$ is tempered then it follows that $c : [0,1] \rightarrow \Omega$ d-Lipschitz curve is equivalent with c $\bar{d}$-Lipschitz curve.

By using the same arguments as in the proof of theorem 35, we get the following extension of proposition 2.4 [13].

Proposition 42. *If* $(\Omega, \bar{d}, \bar{\delta})$ *is tempered, with the Radon-Nikodym property, and* $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ *then*

$$d(x,y) \ = \ \inf \left\{ \int_a^b \phi_d(c(t), \dot{c}(t)) \ dt \ : \ c : [a,b] \to X \ \bar{d}\text{-Lipschitz} \ , \right.$$
$$\left. c(a) = x, c(b) = y \right\}$$

The next theorem is a generalization of the implication (i) $\Rightarrow$ (iii), theorem 3.1 [13].

Theorem 43. *Let* $(\Omega, \bar{d}, \bar{\delta})$ *be a strong dilatation structure which is tempered, with the Radon-Nikodym property, and* $d_n \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$ *a sequence of distances converging to* $d \in \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. *Denote by* L_n, L *the length functional induced by the distance* d_n, *respectively by* d. *Then* L_n Γ*-converges to* L.

Proof. The proof ([13] p. 252-253) is almost identical, we only need to replace everywhere expressions like $\mid x - y \mid$ by $\bar{d}(x,y)$ and use proposition 42, relations (15) and (14) instead of respectively proposition 2.4 and relations (2.6) and (2.3) [13]. $\square$

Using this result we obtain a large class of examples of length dilatation structures.

Corollary 44. *If* $(\Omega, \bar{d}, \bar{\delta})$ *is a strong dilatation structure which is tempered and it has the Radon-Nikodym property then it is a length dilatation structure.*

Proof. Indeed, from the hypothesis we deduce that $\bar{\delta}^x_\varepsilon \bar{d} \ \in \ \mathcal{D}(\Omega, \bar{d}, \bar{\delta})$. For any sequence $\varepsilon_n \to 0$ we thus obtain a sequence of distances $d_n \ = \ \bar{\delta}^x_{\varepsilon_n} \bar{d}$ converging to $\bar{d}^x$. We apply now theorem 43 and we get the result. $\square$

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